QUADRATIC POLYNOMIALS AND QUADRATIC FORMS

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1. Introduction

Let $g(n)=an^2+bn+c$ be a polynomial with integral coefficients ($a>0$) and discriminant $D=b^2-4ac<-4$. We shall be concerned with the problem of showing that, for φ a binary quadratic form satisfying certain natural conditions, the number of $n\leqslant X$ for which $g(n)$ is represented by φ has the expected order of magnitude. The method involves an injection into the half dimensional sieve of a combination of ideas due largely to Chen [1] and Hooley [3]. A sketch of the proof is given in section 2.

For m an integer let $\varrho(m)$ denote the number of solutions of the congruence

$$g(n)\equiv 0 \pmod m.$$

Let P denote a set of primes satisfying:

$$0\leqslant\varrho(p)<p, \tag{1.1}$$

For some fixed K, and all $z\geqslant 2$,

$$\left|\sum_{\substack{p\leqslant z\\ p\in P}}\frac{\varrho(p)}{p-\varrho(p)}\log p-\tfrac{1}{2}\log z\right|\leqslant K. \tag{1.2}$$

For $z\geqslant 2$, we let $P(z)=\prod_{\substack{p<z\\ p\in P}} p$, $\mathcal{A}=\{g(n)\,|\,n\leqslant X\}$ and

$$S(\mathcal{A},P,z)=\sum_{\substack{m\in\mathcal{A}\\ (m,P(z))=1}}1.$$

THEOREM 1. *There exists a positive δ, depending on a, b, c and K such that*

$$S(\mathcal{A},P,z)>\delta X\prod_{\substack{p<z\\ p\in P}}\left(1-\frac{\varrho(p)}{p}\right)-6\pi(z). \tag{1.3}$$