SEPARATRICES AT SINGULAR POINTS OF PLANAR VECTOR FIELDS

BY

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§ 1. Introduction

Let (0, 0) be an isolated singular point of the real analytic vector field

$$\begin{aligned} \dot{x} &= X_d(x, y) + X_{d+1}(x, y) + \ldots = X(x, y) \\ \dot{y} &= Y_d(x, y) + Y_{d+1}(x, y) + \ldots = Y(x, y) \end{aligned} \tag{1.1}$$

where X_i, Y_i are homogeneous of degree i and $X_d^2 + Y_d^2 \not\equiv 0$. We will call the integer $d \geq 1$ the *degree* of the singularity at (0, 0). If (0, 0) is neither a center nor a focus, then a small enough neighborhood of (0, 0) can be decomposed into a finite number of elliptic, hyperbolic and parabolic sectors (for precise definitions and a proof see, for example, [1], [2] or [4]). Bendixson [2] noticed that each hyperbolic sector must contain a branch of $xX + yY = 0$ and each elliptic sector must contain a branch of $X = 0$. He concluded that there are at most $2d + 2$ hyperbolic and $2d$ elliptic sectors. By a *separatrix* at (0, 0) we mean a solution curve of (1.1) that is the boundary of a hyperbolic sector at (0, 0). Since each hyperbolic sector has two boundaries, there are at most $4d + 4$ separatrices at (0, 0). The main result of this paper, proved in Section 3, is that the number of separatrices at (0, 0) is actually bounded by four if $d = 1$, six if $d = 2$, and $4d - 4$ if $d \geq 3$. We give examples to show that these bounds are sharp.

Our result is proved by repeatedly blowing up the singularity (0, 0) of (1.1), a technique that goes back to Bendixson [2] and has been used by many authors (e.g. [1], [3], [5], [8], [9]). We review blowing up in Section 2. Most recent uses of blowing up focus on classifying degenerate singularities of low codimension (i.e., not *too* degenerate). By contrast, we use the technique to determine the most degenerate behavior that can occur at a singularity of given degree. To accomplish this we use counting arguments to keep track of what happens as we repeatedly blow up; these arguments seem to be new.