ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER: IV. THE GENERAL EQUATION $\ddot{y}+kf(y)\dot{y}+g(y)=bkp(\varphi),\quad \varphi=t+\alpha$

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§ 1. We enter now on our complete account of the more general equation

$$\ddot{y}+kf(y)\dot{y}+g(y)=bkp(\varphi),\quad \varphi=t+\alpha.$$

The functions f, g, p are fixed, b is non-negative, and k is large and positive. We proceed to state the long list of assumptions about f, g, p. It may help towards easier reading to imagine that f and g are polynomials and p a trigonometrical polynomial: in so far as hypotheses about the smoothness of f, g, p are concerned our arguments are not essentially different from what they would then be, and the reader may trust us to have taken care of the details. He may similarly take on trust details about the constants connected with these functions, and the various appeals to the f, g, p dictionary (§ 3) that occur in the arguments.

p has continuous p'', is periodic with period normalized to 2π, has mean value 0, and is skew-symmetric, i.e. $p(\pi+\varphi)=-p(\varphi)$. Any integral $\int p\,d\varphi$ is periodic; we define $p_1(\varphi)$ be that one for which the mean value is 0. It also is skew-symmetric. It is now an essential assumption that p_1 *attains its upper* (*and consequently also its lower*) *bound once only in a period*. We normalize p to make 1 the upper bound of p_1, to be attained at $\frac{1}{2}\pi$. So $p_1(\frac{1}{2}\pi)=-p_1(-\frac{1}{2}\pi)=1$, $p(\pm\frac{1}{2}\pi)=0$. $p'(-\frac{1}{2}\pi)$ is non-negative; we suppose it a *positive* constant a_2.

$f(y)$ is even, with continuous f''. It has a single pair of zeros, normalized to ± 1; $f'(1)$ is a *positive* constant a_1, and f has a positive lower bound in (say) $y\geq 2$. We define $F(y)=\int_0^y f(y)\,dy$; F is odd. We normalize f to make $F(-1)$ (certainly positive) $\frac{2}{3}$. This will make $\frac{2}{3}$ the critical value of b, as for van der Pol's equation; the behaviour for $b>\frac{2}{3}$ is crude, and we suppose for simplicity that $0<b<2$ as before.