ON THE ROOTS OF THE RIEMANN ZETA-FUNCTION

BY

D. H. LEHMER

in Berkeley, California, U.S.A.

It is the purpose of this paper to give an account of numerical calculations relating to the behavior of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s = \sigma + i t)$$

on the critical line $\sigma = {}^1/_2$, $t > 0$. These results confirm those made previously by Gram [1], Hutchinson [2], Titchmarsh [3] and Turing [4] and extend these to the first 10,000 zeros of $\zeta(s)$. All these zeros have real parts equal to one half and are simple. Thus the Riemann Hypothesis is true at least for $t \leq 9878.910$. This extension of our knowledge of $\zeta({}^1/_2 + i t)$ is made possible by the use of the electronic computer known as the SWAC while it was the property of the United States National Bureau of Standards. Actually only a few hours of machine time was needed and much more could be done along the same lines by this or any other really high speed computer.

A brief history of previous results and contemplated calculations may be given as follows. The work of J. P. Gram (in 1902–4) was largely for real s. However, he gave the first ten roots of $\zeta(s)$ to 6 decimals and five further ones with less accuracy. He is also to be credited with a valuable observation, now known as Gram's Law, which may be stated as follows. Let n be a positive integer and let τ_n be the real positive root of the equation

$$\pi^{-1} \operatorname{Im} (\log \Gamma ({}^1/_4 + \pi i \tau)) - \tau \log \pi = n.$$

We call τ_n the nth Gram point and the interval

$$I_n : (\tau_n, \tau_{n+1})$$

the nth Gram interval. Gram's Law states that $\zeta({}^1/_2 + 2\pi i \tau)$ has a single root in I_n. This law implies the Riemann Hypothesis and the verification of the latter depends