RIEMANNIAN GEOMETRY AS DETERMINED BY THE VOLUMES OF SMALL GEODESIC BALLS

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1. Introduction

Let M be an n-dimensional Riemannian manifold of class C^∞. For small $r>0$ let $V_m(r)$ denote the volume of a geodesic ball with center m and radius r. This paper is concerned with the following question: *To what extent do the functions $V_m(r)$ determine the Riemannian geometry of M?* In particular we shall be concerned with the following conjecture:

(I) *Suppose*

$$V_m(r) = \omega r^n \tag{1.1}$$

for all $m \in M$ and all sufficiently small $r>0$. Then M is flat.

(Here ω = the volume of the unit ball in $\mathbf{R}^n$. The simplest expression for ω is $\omega = (1/(\frac{1}{2}n)!)\pi^{n/2}$ where $(\frac{1}{2}n)! = \Gamma(\frac{1}{2}n+1)$.)

First we make several remarks.

1. Our method for attacking the conjecture (I) will be to use the power series expansion for $V_m(r)$. This expansion will be considered in detail in section 3; however, the general facts about it are the following: (a) the first term in the series is ωr^n; (b) the coefficient of r^{n+k} vanishes provided k is odd; (c) the coefficients of r^{n+k} for k even can be expressed in terms of curvature. Unfortunately the nonzero coefficients depend on curvature in a rather complicated way, and this is what makes the resolution of the conjecture (I) an interesting problem.

2. To our knowledge the power series expansion for $V_m(r)$ was first considered in 1848 by Bertrand–Diguet–Puiseux [6]. See also [14, p. 209]. In these papers the first two terms of the expansion for $V_m(r)$ are computed for surfaces in $\mathbf{R}^3$:

$$V_m(r) = \pi r^2 \left\{1 - \frac{K}{12} r^2 + O(r^4)\right\}_m, \tag{1.2}$$

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