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New constructions of fundamental polyhedra in complex hyperbolic space

by

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1. Introduction

Construction of lattices in $\mathrm{PU}(n,1)$ has been a major challenge in the last decades. In particular, in contrast with the situation in real hyperbolic space, non-arithmetic lattices have been found only in dimensions up to three (see [Mo1], [DM] and [Mo2]).

Mostow's first examples in $\mathrm{PU}(2,1)$ were constructed by giving explicit generators, and verifying that the corresponding groups are discrete by finding a fundamental domain for their action. In complex hyperbolic space, or in any space where sectional curvature is not constant, such an approach is bound to be at least somewhat difficult since there are no totally geodesic real hypersurfaces.

Other direct proofs of discreteness have led to domains bounded by various types of hypersurfaces, each of them adapted to the situation at hand (see the constructions in [FPk], [S1] and [S2]). There is a canonical construction due to Dirichlet, where the boundary of the domain is made up of bisectors, i.e. hypersurfaces equidistant from two given points. One chooses a point p_0, and considers the set F of points closer to p_0 than to any other point in its orbit under the group. It is obvious that the group is discrete if and only if F contains a neighborhood of p_0, but the set F is in general very difficult to study or describe. Such a description amounts to solving a system of infinitely many quadratic inequalities in four variables (the real and imaginary part of the coordinates in the complex 2-ball). In particular, bisector intersections are neither transverse nor connected in general.

Nevertheless, this was the original approach taken by Mostow ([Mo1]) to study a remarkable class of groups, $\Gamma(p,t)$ (where $p=3,4,5$ and t is a real parameter). Each of these is generated by three braiding complex reflections, R_1, R_2 and R_3, of order p; it is contained with finite index in the group $\widetilde{\Gamma}(p,t)$ generated by R_1 and the elliptic element J which conjugates R_i into R_{i+1} (see §2.2 for further details).