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Julia–Fatou–Sullivan theory for real one-dimensional dynamics

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Introduction

Our aim is to show that the Julia–Fatou–Sullivan structure theory for the dynamics of rational maps is also valid for smooth endomorphisms of the circle (and of the interval) under extremely mild smoothness and non-flatness conditions.

In order to stress the similarity between real and complex one-dimensional dynamics let us recall the main results from the Fatou–Julia–Sullivan theory. If f is a rational map then there is a dynamical decomposition of the Riemann sphere into the disjoint union of two totally invariant (i.e., both forward and backward invariant) sets $J(f), F(f)$. Here, $F(f)$ is the domain of normality of the family of iterates of f, and is called the Fatou set. Its complement, which is called the Julia set of f, is a compact set, which contains all the complications of the dynamics of f. The connected components of the open set $F(f)$ are mapped onto each other by f. Hence the orbit of a component of $F(f)$ is the union of some components of $F(f)$. Julia proved in the beginning of the century that if a component of $F(f)$ is periodic and contains an attracting periodic point then the orbit of this component must contain a critical point. Sullivan, in the remarkable paper [Su], showed via quasi-conformal deformations that the components are eventually periodic and fall into finitely many orbits.

Let N be either the circle S^1 or a compact interval of the real line and $f: N \to N$ be a smooth endomorphism. A *critical point* of f is a point where the derivative vanishes. A critical point is *non-flat* if some (higher) derivative is non-zero. A critical point is an *inflection point* if it has a neighbourhood where f is monotone. Otherwise it is called a *turning point*. Assume f is not a homeomorphism (if f is a homeomorphism then these maps correspond to degree ± 1 rational maps of the Riemann sphere and the situation is

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