On the model companion of the theory of e-fold ordered fields

by

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0. Introduction

The present work is inspired by three papers, [11] of Van den Dries, [9] of Prestel and [5]. Van den Dries considers structures of the form $(K, P_1, \ldots, P_e)$, where K is a field and $P_1, \ldots, P_e$ are e orderings of the K. They are called, *e-fold ordered fields*. The appropriate first ordered language is denoted by $\mathscr{L}_e$. He proves that the theory of e-fold ordered fields in $\mathscr{L}_e$ has a model companion $\overline{OF}_e$. The models $(K, P_1, \ldots, P_e)$ of $\overline{OF}_e$ are characterized on one hand by being existentially closed in the family of e-fold, ordered fields, and by satisfying certain axioms of $\mathscr{L}_e$ on the other hand.

In particular Van den Dries proves that the absolute Galois group $G(K)$ of K is a pro-2-group generated by e involutions. If K is algebraic over $\mathbf{Q}$ and R is a real closure of $\mathbf{Q}$, this implies that there exist $\sigma_1, \ldots, \sigma_e \in G(\mathbf{Q})$ such that $K = R^{\sigma_1} \cap \ldots \cap R^{\sigma_e}$. In general, if $\sigma_1, \ldots, \sigma_e \in G(\mathbf{Q})$, we write $\mathbf{Q}_\sigma = R^{\sigma_1} \cap \ldots \cap R^{\sigma_e}$ and denote by $P_{\sigma i}$ the ordering of $\mathbf{Q}$ induced by the unique ordering of the real closed field R^{σ_i}. In this way we attain a family of e-fold ordered fields, $\mathscr{Q}_\sigma = (\mathbf{Q}_\sigma, P_{\sigma 1}, \ldots, P_{\sigma e})$, indexed by $G(\mathbf{Q})^e$.

Geyer proves in [4] that for almost all $\sigma \in G(\mathbf{Q})^e$ (in the sense of the Haar measure of $G(\mathbf{Q})^e$), the group $G(\mathbf{Q}_\sigma)$ is isomorphic to the free product, $\hat{D}_e$, of e copies of $\mathbf{Z}/2\mathbf{Z}$, in the category of profinite groups. This takes us away from the models of $\overline{OF}_e$ and leads us in [5] to make the following

Definition. An e-fold ordered field $(K, P_1, \ldots, P_e)$ is said to be a *Geyer-field of corank e* if the following conditions hold:

(α) If V is an absolutely irreducible variety defined over K and if each of the orderings P_i extends to the function field of V, then V has a K-rational point.

(β) The orderings $P_1, \ldots, P_e$ induce distinct topologies on K.

(γ) We have $G(K) \cong \hat{D}_e$.

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