Harmonic volumes

by

BRUNO HARRIS[1]

Brown University Providence, RI, U.S.A.

1. Introduction

Let X be a compact Riemann surface and dh_i, $i=1,2,3$, three real harmonic 1-forms on X satisfying

$$\int_X dh_i \wedge dh_j = 0, \quad i,j = 1,2,3, \tag{1.1}$$

$$\int_\gamma dh_i \in \mathbf{Z}, \quad \text{for any 1-cycle } \gamma \text{ on } x. \tag{1.2}$$

To this triple of harmonic 1-forms, we associate a point in $\mathbf{R}/\mathbf{Z}$, denoted $I(dh_1, dh_2, dh_3)$, which can be defined in two equivalent ways: first, as an iterated integral

$$\int_\gamma (h_1\, dh_2 - \eta_{12}) \bmod \mathbf{Z}, \tag{1.3}$$

where γ is a path on X Poincaré-dual to the cohomology class of dh_3, h_1 is a function on γ obtained by integrating dh_1, and η_{12} is a 1-form on X satisfying $d\eta_{12} = dh_1 \wedge dh_2$ (and orthogonal to all closed 1-forms); second, as a volume mod $\mathbf{Z}$: namely by (1.2), we can integrate the dh_i on X to obtain h_i: $X \to \mathbf{R}/\mathbf{Z}$ which are harmonic. Then $h = (h_1, h_2, h_3)$: $X \to \mathbf{R}^3/\mathbf{Z}^3 = T^3$, and it follows easily from (1.1) that $h(X)$, regarded as a singular 2-cycle, bounds a singular 3-chain c_3 (unique mod integral 3-cycles); we can then take the volume of c_3 (mod $\mathbf{Z}$) to define $I(dh_1, dh_2, dh_3)$: we call this a "harmonic volume".

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