SPECTRAL SYNTHESIS IN SOBOLEV SPACES, AND UNIQUENESS OF SOLUTIONS OF THE DIRICHLET PROBLEM

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1. Introduction

Consider functions f in the Sobolev space $W_m^q(\mathbf{R}^d)$, $1<q<\infty$, i.e. functions such that $\sum_{0\leqslant|\alpha|\leqslant m}\int_{\mathbf{R}^d}|D^\alpha f|^q dx=\|f\|_{m,q}^q<\infty$. For any set E in $\mathbf{R}^d$ one can define the trace on E of f and of its partial derivatives $D^\alpha f$, $|\alpha|\leqslant m-1$, in a natural way. (See Section 2.) We denote these traces by $f|_E$ and $D^\alpha f|_E$. Our main result is the following theorem.

THEOREM 1.1. *Let $f\in W_m^q(\mathbf{R}^d)$ for some $q>2-1/d$, and some positive integer m. Let $K\subset\mathbf{R}^d$ be closed, and suppose that $D^\alpha f|_K=0$ for all α, $0\leqslant|\alpha|\leqslant m-1$. Then $f\in \overset{0}{W}{}_m^q(K^c)$, i.e. there exist functions $\varphi_n\in C_0^\infty$ such that each φ_n vanishes on a neighborhood of K, and $\lim_{n\to\infty}\|f-\varphi_n\|_{m,q}=0$.*

By analogy with the classical spectral synthesis of Beurling (see e.g. [20]) we say that sets K with the approximation property in the theorem admit *(m, q)-synthesis*. Thus, in contrast to the situation in harmonic analysis, the conclusion here is that *all closed sets in $\mathbf{R}^d$ admit (m, q)-synthesis, at least if $q>2-1/d$.*

Among the consequences we mention the following uniqueness theorem for the Dirichlet problem. This is in fact an equivalent formulation of the result in the case $q=2$. By way of illustration we only formulate the theorem in the simplest case. Generalizations to more general elliptic equations are immediate. See T. Kolsrud [21] for an extension to situations where u is defined only in G.

THEOREM 1.2. *Let $G\subset\mathbf{R}^d$ be a bounded open set. Let $u\in W_m^2(\mathbf{R}^d)$ satisfy $\Delta^m u=0$ in G, and $D^\alpha u|_{\partial G}=0$, $0\leqslant|\alpha|\leqslant m-1$. Then $u\equiv 0$ in G.*

That this is a consequence of Theorem 1.1 is obvious, because it is well known that if $\Delta^m u=0$ in G and $u\in \overset{0}{W}{}_m^2(G)$, then $u=0$.

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