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Solving the quintic by iteration

by

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1. Introduction

According to Dickson, Euler believed every algebraic equation was solvable by radicals [2]. The quadratic formula was known to the Babylonians; solutions of cubic and quartic polynomials by radicals were given by Scipione del Ferro, Tartaglia, Cardano and Ferrari in the mid-1500s. Abel's proof of the insolvability of the general quintic polynomial appeared in 1826 [1]; later Galois gave the exact criterion for an equation to be solvable by radicals: its Galois group must be solvable. (For a more complete historical account of the theory of equations, see van der Waerden [20], [21].)

In this paper, we consider solving equations using *generally convergent purely iterative algorithms*, defined by Smale [17]. Such an algorithm assigns to its input data v a rational map $T_v(z)$, such that $T_v^n(z)$ converges for almost all v and z; the limit point is the *output* of the algorithm.

This context includes the classical theory of solution by radicals, since nth roots can be reliably extracted by Newton's method.

In [12] a rigidity theorem is established that implies the maps $T_v(z)$ for varying v are all conformally conjugate to a fixed model $f(z)$. Thus the Galois theory of the output of T must be implemented by the conformal automorphism group $\mathrm{Aut}(f)$, a finite group of Möbius transformations.

The classification of such groups is well-known: $\mathrm{Aut}(f)$ is either a cyclic group, dihedral group, or the group of symmetries of a regular tetrahedron, octahedron or

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