CONFORMAL INVARIANTS AND FUNCTION-THEORETIC NULL-SETS.

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§ 1. Introduction.

The most useful conformal invariants are obtained by solving conformally invariant extremal problems. Their usefulness derives from the fact that they must automatically satisfy a principle of majorization. There is a rich variety of such problems, and if we would aim at completeness this paper would assume forbidding proportions. We shall therefore limit ourselves to a few particularly simple invariants and study their properties and interrelations in considerable detail.

Each class of invariants is connected with a category of null-sets, which by this very fact enter naturally in function-theoretic considerations. A null-set is the complement of a region for which a certain conformal invariant degenerates. Inequalities between invariants lead to inclusion relations between the corresponding classes of null-sets.

Throughout this paper Ω will denote an open region in the extended z plane, and z_0 will be a distinguished point in Ω. Most results will be formulated for the case $z_0 \neq \infty$, but the transition to $z_0 = \infty$ is always trivial. In some instances the latter case offers formal advantages.

We shall consider classes of functions $f(z)$ which are analytic and single-valued in some region Ω. For a general class $\mathfrak{F}$ the region Ω is allowed to vary with f, but the subclass of functions in a fixed region Ω will be denoted by $\mathfrak{F}(\Omega)$. For $z_0 \in \Omega$ we introduce the quantity

$$M_{\mathfrak{F}}(z_0, \Omega) = \sup_{f \in \mathfrak{F}(\Omega)} |f'(z_0)|. \tag{1}$$