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Beurling's Theorem for the Bergman space

by

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1. Introduction

Many interesting Hilbert space operators can be modelled by natural operations on spaces of functions analytic in the unit disk $\mathbf{D}$. The most basic of these operations is multiplication by the coordinate function z, and in this case the invariant subspaces of the operator correspond to what we call the invariant subspaces of the function space, i.e. those closed subspaces M for which $zM\subset M$. As a matter of terminology, we will call the smallest invariant subspace containing a given set S the *invariant subspace generated by* S, and we will denote it by $[S]$. An invariant subspace generated by a single function will be called *cyclic.*

The best known example in this area is the case where the function space is the Hardy space H^2. This space consists of those functions f analytic in $\mathbf{D}$ for which

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi} < \infty.$$

By means of radial limits, H^2 can be identified with the subspace of $L^2(\partial\mathbf{D})$ of functions f for which

$$\hat{f}(n) = \int_{|z|=1} f(z)\,\bar{z}^n \, \frac{|dz|}{2\pi} = 0 \quad \text{for } n = -1, -2, \dots .$$

Multiplication by z on H^2 models the unilateral shift $(a_0, a_1, \dots)\mapsto(0, a_0, a_1, \dots)$ on l^2_+, an operator of basic importance in many areas of analysis. A famous classical result of A. Beurling [B] classifies the invariant subspaces of H^2, and thus the invariant subspaces of the unilateral shift. To describe this result we recall that an *inner function* in H^2 is a function $\varphi\in H^2$ whose radial limits have modulus 1 a.e. on $\partial\mathbf{D}$. We will use the notation $M\ominus N = M\cap N^{\perp}$ for closed subspaces N, M such that $N\subset M$.

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