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On the p-typical curves in Quillen's K-theory

by

LARS HESSELHOLT

Massachusetts Institute of Technology
Cambridge, MA, U.S.A.

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Introduction

Twenty years ago Bloch, [Bl], introduced the complex $C_*(A;p)$ of p-typical curves in K-theory and outlined its connection to the crystalline cohomology of Berthelot–Grothendieck. However, to prove this connection Bloch restricted his attention to the symbolic part of K-theory, since only this admitted a detailed study at the time. In this paper we evaluate $C_*(A;p)$ in terms of the fixed sets of Bökstedt's topological Hochschild homology. Using this we show that for any smooth algebra A over a perfect field k of positive characteristic, $C_*(A;p)$ is isomorphic to the de Rham–Witt complex of Bloch–Deligne–Illusie. This confirms the outlined relationship between p-typical curves in K-theory and crystalline cohomology in the smooth case. In the singular case, however, we get something new. Indeed, we calculate $C_*(A;p)$ for the ring $k[t]/(t^2)$ of dual numbers over k and show that in contrast to crystalline cohomology, its cohomology groups are finitely generated modules over the Witt ring $W(k)$.

Let A be a ring, by which we shall always mean a commutative ring, and let $K(A)$ denote the algebraic K-theory spectrum of A. More generally, if $I\subset A$ is an ideal, $K(A,I)$ denotes the relative algebraic K-theory, that is, the homotopy theoretical fiber of the map $K(A)\to K(A/I)$. We define the *curves on* $K(A)$ to be the homotopy limit of spectra

$$C(A)=\underset{\underset{n}{\longleftarrow}}{\mathrm{holim}}\,\Omega K(A[X]/(X^n),(X)).$$