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Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions

by

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1. Introduction

1.1. The operator $\partial_1^3+\partial_2^3+\partial_3^3$ was considered—to my knowledge—for the first time in 1913 in N. Zeilon's article [20], wherein he generalizes I. Fredholm's method of construction of fundamental solutions (see [5]) from homogeneous *elliptic* equations to arbitrary homogeneous equations in three variables with a *real-valued* symbol (cf. [20, II, pp. 14–22], [6, Chapter 11, pp. 146–148]). An explicit formula for a fundamental solution was given in [19]. The objective of this paper is to generalize the calculations in [19] to the operators $\partial_1^3+\partial_2^3+\partial_3^3+3a\partial_1\partial_2\partial_3$, $a\in\mathbf{R}\setminus\{-1\}$. As discussed below, this class of operators comprises all real homogeneous cubic operators of principal type in three dimensions.

According to Newton's classification of real elliptic curves, the non-singular real homogeneous polynomials $P(\xi)$ of third order in three variables are divided into two types according to whether the real projective curve $\{[\xi]\in\mathbf{P}(\mathbf{R}^3):P(\xi)=0\}$ consists of one or of two connected components, respectively. (For $\xi\in\mathbf{R}^n\setminus\{0\}$, $[\xi]\in\mathbf{P}(\mathbf{R}^n)$ denotes the corresponding projective point, i.e., the line $\{t\xi:t\in\mathbf{R}\}$.) In Hesse's normal form, all non-singular real cubic curves are—up to linear transformations—given by

$$P_a(\xi)=\xi_1^3+\xi_2^3+\xi_3^3+3a\xi_1\xi_2\xi_3,\quad a\in\mathbf{R}\setminus\{-1\}$$

(cf. [3, 7.3, Satz 4, p. 379; English transl., p. 293], [4, §7, (10), p. 39], [17, §1.4, p. 19]). Let $X_a:=\{[\xi]\in\mathbf{P}(\mathbf{R}^3):P_a(\xi)=0\}$ denote the real projective variety defined by P_a. For $a>-1$, X_a is connected, whereas, for $a<-1$, X_a consists of two components (cf. Figure 1). The corresponding operators $P_a(\partial)$ also differ from the physical viewpoint: For $a<-1$, every projective line through $[1,1,1]$ intersects X_a in three different projective points, and thus P_a is strongly hyperbolic in the direction $(1,1,1)$ ([1, 3.8, p. 129]); for $a>-1$, P_a is not hyperbolic in any direction, nor is it an evolution operator (cf. [15, Example 1, p. 463] for the case of $a=0$).