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The zero multiplicity of linear recurrence sequences

by

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1. Introduction

A *linear recurrence sequence of order* t is a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying a relation

$$u_n = c_1 u_{n-1} + \ldots + c_t u_{n-t} \quad (n \in \mathbb{Z}) \tag{1.1}$$

with $t>0$ and fixed coefficients $c_1, \ldots, c_t$, but no relation with fewer than t summands, i.e., no relation $u_n = c'_1 u_{n-1} + \ldots + c'_{t-1} u_{n-t+1}$. This implies in particular that the sequence is not the zero sequence, and that $c_t \neq 0$. The *companion polynomial* of the relation (1.1) is

$$\mathcal{P}(z) = z^t - c_1 z^{t-1} - \ldots - c_t.$$

Write

$$\mathcal{P}(z) = \prod_{i=1}^{k} (z-\alpha_i)^{a_i} \tag{1.2}$$

with distinct roots $\alpha_1, \ldots, \alpha_k$. The sequence is said to be *nondegenerate* if no quotient α_i/α_j $(1 \leqslant i < j \leqslant k)$ is a root of 1. The *zero multiplicity* of the sequence is the number of $n \in \mathbb{Z}$ with $u_n = 0$. For an introduction to linear recurrences and exponential equations, see [10].

A classical theorem of Skolem–Mahler–Lech [4] says that a nondegenerate linear recurrence sequence has finite zero multiplicity. Schlickewei [6] and van der Poorten and Schlickewei [5] derived upper bounds for the zero multiplicity when the members of the sequence lie in a number field K. These bounds depended on the order t, the degree of K, as well as on the number of distinct prime ideal factors in the decomposition of the fractional ideals (α_i) in K. More recently, Schlickewei [7] gave bounds which depend

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