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The concentration function of additive functions on shifted primes

by

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In memory of Rolando Chaqui

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A real valued function f defined on the positive integers is *additive* if it satisfies $f(rs)=f(r)+f(s)$ whenever r and s are coprime. Such functions are determined by their values on the prime-powers.

For an additive arithmetic function f, let C_h denote the frequency amongst the integers n not exceeding x of those for which $h<f(n)\leqslant h+1$. Estimates for C_h that are uniform in h, f and x play a vital rôle in the study of the value distribution of additive functions. They can be employed to develop criteria necessary and sufficient that a suitably renormalised additive function possess a limiting distribution, as well as to elucidate the resulting limit law. They bear upon problems of algebraic nature, such as the product and quotient representation of rationals by rationals of a given type. In that context their quantitative aspect is important.

It is convenient to write $a\ll b$ *uniformly in* α if on the values of α being considered, the functions a, b satisfy $|a(\alpha)|\leqslant cb(\alpha)$ for some absolute constant c. When the uniformity is clear, I do not declare it.

Let

$$W(x)=4+\min_{\lambda}\left(\lambda^2+\sum_{p\leqslant x}\frac{1}{p}\min(1,|f(p)-\lambda\log p|)^2\right),$$

where the sum is taken over prime numbers. Improving upon an earlier result of Halász, Ruzsa proved that $C_h\ll W(x)^{-1/2}$, uniformly in h, f and $x\geqslant 2$ [15]. This result is best possible in the sense that for each of a wide class of additive functions there is a value of h so that the inequality goes the other way.