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Subgroup growth of lattices in semisimple Lie groups

by

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1. Introduction

Let H be a simple real Lie group; thus H is the connected part of $G(\mathbf{R})$ for some simple algebraic group G. Let K be a maximal compact subgroup of H, $X=H/K$ be the associated symmetric space, and let Γ be a lattice in H, i.e., a discrete subgroup of finite covolume in H. The lattice Γ is said to be *uniform* if H/Γ is compact, and *non-uniform* otherwise. We denote by $s_n(\Gamma)$ the number of subgroups of Γ of index at most n. The study of $s_n(\Gamma)$ for finitely generated groups Γ has been a focus of a lot of research in the last two decades (see [LuS] and the references therein). Our first result is a precise (and somewhat surprising) estimate of $s_n(\Gamma)$ for higher-rank lattices.

THEOREM 1. *Assume that* $\mathbf{R}$-rank$(H)\geqslant 2$ *and H is not locally isomorphic to* $D_4(\mathbf{C})$. *Then for every non-uniform lattice Γ in H, the limit*

$$\lim_{n\to\infty} \frac{\log s_n(\Gamma)}{(\log n)^2/\log\log n}$$

exists and equals a constant $\gamma(H)$ which depends only on H and not on Γ. The number $\gamma(H)$ is an invariant which is easily computed from the root system of G.

The theorem shows that different lattices in the same Lie group have some hidden algebraic similarity; a phenomenon which also presents itself as a corollary of Margulis super-rigidity, which implies that H can be reconstructed from each Γ.

Every conjugacy class of subgroups of Γ of index n has size at most n (which is negligible compared to $s_n(\Gamma)$) and defines a unique cover of the Riemannian manifold $M=\Gamma\backslash X$. Hence Theorem 1 is equivalent to the following theorem.

THEOREM 1′. *With the same assumptions on H as in Theorem* 1. *Let M be a non-compact manifold of finite volume covered by X, and let $b_n(M)$ be the number of*