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# Counting congruence subgroups

by

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## 0. Introduction

Let $k$ be an algebraic number field, $\mathcal{O}$ its ring of integers, $S$ a finite set of valuations of $k$ (containing all the archimedean ones), and $\mathcal{O}_S=\{x\in k \mid v(x)\geqslant 0 \text{ for all } v\notin S\}$. Let $G$ be a semisimple, simply-connected, connected algebraic group defined over $k$ with a fixed embedding into $\mathrm{GL}_d$. Let $\Gamma=G(\mathcal{O}_S)=G\cap\mathrm{GL}_d(\mathcal{O}_S)$ be the corresponding $S$-arithmetic group. We assume that $\Gamma$ is an infinite group (equivalently, $\prod_{\nu\in S} G(k_\nu)$ is not compact).

For every non-zero ideal $I$ of $\mathcal{O}_S$ let

$$\Gamma(I)=\mathrm{Ker}(\Gamma\to\mathrm{GL}_d(\mathcal{O}_S/I)).$$

A subgroup of $\Gamma$ is called a congruence subgroup if it contains $\Gamma(I)$ for some $I$.

The topic of counting congruence subgroups has a long history. Classically, congruence subgroups of the modular group were counted as a function of the genus of the associated Riemann surface. It was conjectured by Rademacher that there are only finitely many congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ of genus zero. Petersson [Pe] proved that the number of all subgroups of index $n$ and fixed genus goes to infinity exponentially as $n\to\infty$. Dennin [De] proved that there are only finitely many congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ of given fixed genus and solved Rademacher's conjecture. A quantitative result was proved by Thompson [T] and Cox–Parry [CP] who showed (among other interesting results) that

$$\lim \frac{\mathrm{genus}(\Lambda)}{[\mathrm{SL}_2(\mathbf{Z}):\Lambda]}=\frac{1}{12},$$

where the limit goes over congruence subgroups $\Lambda$ of $\mathrm{SL}_2(\mathbf{Z})$ with index going to $\infty$. It does not seem possible, however, to accurately count all congruence subgroups of index at most $r$ in $\mathrm{SL}_2(\mathbf{Z})$ by using the theory of Riemann surfaces of fixed genus.

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