Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group

by

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1. Introduction and preliminaries

1 A. As is well-known, one can always extend a Möbius transformation of $\bar{\mathbf{R}}^n$ $(=\mathbf{R}^n \cup \{\infty\})$ to a Möbius transformation of the hyperbolic $(n+1)$-space $H^{n+1}=\{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}: x_{n+1}>0\}$. For instance, this can be done as follows. Let $z \in H^{n+1}$. Pick a triple $x=(u, v, w) \in (\bar{\mathbf{R}}^n)^3$ of distinct points such that z is on the hyperbolic line L with endpoints u and v, and such that the hyperbolic ray R with endpoints z and w intersects L orthogonally. Then we write

$$z = p(u, v, w) = p(x). \tag{1.1}$$

If now g is a Möbius transformation of $\bar{\mathbf{R}}^n$, then the extension of g to H^{n+1} is given by

$$g(z) = p(g(u), g(v), g(w)) = pg(x). \tag{1.2}$$

If g is a Möbius transformation, then (1.2) is independent of the choice of the triple satisfying (1.1), but this is not true of non-Möbius g. However, and this observation started this paper, if g is quasiconformal, then (1.2) defines a kind of fuzzy image of z for $z \in H^{n+1}$ which satisfies a certain type of Lipschitz condition. We explain this now in more detail.

First, if two triples $x, x' \in p^{-1}(z)$, then the hyperbolic distance

$$d(pg(x), pg(x')) \leq m, \tag{1.3}$$

where $m \geq 0$ depends only on n and on the dilatation of g (Theorem 3.4). Thus the indeterminacy in the image of z is uniformly bounded for $z \in H^{n+1}$.

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