ON THE NUMBER OF INVARIANT CLOSED GEODESICS

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The study of closed (periodic) geodesics has a long and rich history. After Fet and Lyusternik [2] in 1951 proved that any compact riemannian manifold has at least one closed geodesic, the most outstanding problem has been whether such a manifold has actually infinitely many distinct closed geodesics. Here closed geodesics are always understood to be non-constant, and two geodesics are said to be distinct if one is not a reparametrization of the other. No real progress was made untill 1969 when Gromoll and Meyer [7] obtained the following celebrated result.

THEOREM. *Let M be a compact connected and simply connected riemannian manifold. Then M has infinitely many closed geodesics if the sequence of Betti numbers for the (rational) homology of the space of all maps $S^1 \to M$ is unbounded.*

Here a map is always understood to be continuous and the space of maps $S^1 \to M$ is endowed with the compact-open (uniform) topology. Recently Sullivan and Vigué [21] showed that the topological condition on M in the above theorem is satisfied if and only if the (rational) cohomology ring of M is not generated by one element.

Just very recently we have received the second revised and enlarged edition of a manuscript to a monograph on closed geodesics by W. Klingenberg [13]. In that manuscript a proof for the existence of infinitely many closed geodesics on any 1-connected compact riemannian manifold is offered. The proof involves new methods and ideas and is very complicated.

A related but more general theory than that of closed geodesics is the one of isometry-invariant geodesics developed by the first named author in [8] and [9]. A non-constant geodesic $c\colon \mathbf{R} \to M$ is said to be invariant under an isometry $A\colon M \to M$ if $A(c(t)) = c(t+1)$ for all $t \in \mathbf{R}$. Clearly an A-invariant geodesic with $A = id_M$ is simply a closed geodesic and vice versa. In contrast to the case of closed geodesics, there are examples of isometries

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