SOME EXTREMAL PROBLEMS IN THE THEORY OF NUMERICAL RANGES

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0. Introduction

Let $\mathfrak{A}$ be a complex unital Banach algebra with dual space $\mathfrak{A}'$. The *numerical range* of an element a of $\mathfrak{A}$, $V(\mathfrak{A}, a)$, is defined by

$$V(\mathfrak{A}, a) = \{f(a) \colon f \in \mathfrak{A}', f(1) = 1 = \|f\|\}$$

and is a compact convex subset of the complex field $\mathbf{C}$. The *numerical radius* of a, $v(a)$, is then defined by

$$v(a) = \max \{|\lambda| : \lambda \in V(\mathfrak{A}, a)\}.$$

Wherever possible we shall follow the notation of Bonsall and Duncan [6] to which we refer the reader for a systematic account of the theory of numerical ranges. In this paper we shall consider several problems of the following nature. Suppose that the numerical range $V(\mathfrak{A}, a)$ is restricted in size and shape. What conditions are then implied on the algebra generated by 1 and a; for example, how large can $\|a^n\|$ be? Several results are known in this area. For example, if $v(a) = 1$, then

$$\|a^n\| \leqslant n! \left(\frac{e}{n}\right)^n \quad (n = 1, 2, 3, \ldots)$$

and these inequalities are best possible; see Bollobás [3], [4], Browder [8], Crabb [9], [10]. In particular, the power inequality

$$v(a^n) \leqslant v(a)^n \quad (n = 1, 2, 3, \ldots)$$

which is known to hold in B*-algebras, does not hold for arbitrary Banach algebras. On the other hand, if $V(\mathfrak{A}, a)$ is a subset of the real field $\mathbf{R}$, i.e. if a is *Hermitian*, then

$$\varrho(a) = v(a) = \|a\|$$