NEAR INCLUSIONS OF C^*-ALGEBRAS

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1. Introduction

For C^*-subalgebras A and B of a C^*-algebra C we study the relation $A \overset{\gamma}{\subset} B$, which means that for any a in A, there exists an operator b in B such that $\|a-b\| \leqslant \gamma \|a\|$.

The main reason why we have investigated those relations, is that we think, that if γ is small enough, B must have a subalgebra which shares some of its properties with A, and in turn we hope that we can get information on the space of C^*-subalgebras of a given C^*-algebra.

Our methods yield positive answers in several cases, and we prove under some conditions on A and B that there exists a unitary operator u on a underlying Hilbert space such that u is close to the identity and uAu^* is contained in B, (Th. 4.1, Cor. 4.2, Th. 4.3, Th. 5.3). The theorems in section 4 are, generally speaking, obtained in the situation where A and B are von Neumann algebras on a Hilbert space and one of them is injective.

Theorem 5.3 tells that B contains such a twisted copy of A, if A is finite-dimensional and γ is less than 10^{-4}. In particular one should remark that the result is independent of the dimension of A.

Having the result of section 5 we are able to show in section 6 that if A is the norm closure of an increasing sequence of finite dimensional C^*-algebras (AF for short), A and B satisfy $A \overset{\gamma}{\subset} B$, $B \overset{\gamma}{\subset} A$ and γ is less than 10^{-9}, then B is also AF. This implies that B is unitarily equivalent to A in these cases.

At the end of section 6 we study the relations $A \overset{\gamma}{\subset} B$, $B \overset{\gamma}{\subset} A$ for other types of C^*-algebras, and we find that if A is nuclear and γ is less than 10^{-2} then B is also nuclear and the dual spaces A^* and B^* are isomorphic via a completely positive isometry.

The proofs of the results in the sections 4 and 5 are made in three steps.

Suppose $A \overset{\gamma}{\subset} B$, then the first step is to find a completely positive linear map of A into B which is close to the identity on A. In the case where B is an injective von Neumann