ON THE TRACE FORMULA FOR HECKE OPERATORS

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The formula to be proved in this paper has roughly the following form:

$$\mathrm{tr}\,(\Gamma\alpha\Gamma\,|\,A_m) - \mathrm{tr}\,(\Gamma\alpha^{-1}\Gamma\,|\,B_{2-m}) = \textstyle\sum_C J(C).$$

Here Γ is a discrete subgroup of $SL_2(\mathbf{R})$ such that $SL_2(\mathbf{R})/\Gamma$ is of finite measure, m an arbitrary rational number, A_m the space of cusp forms of weight m with respect to Γ on the upper half complex plane $\mathfrak{H}$, B_{2-m} the space of integral forms of weight $2-m$ with respect to Γ, α an element of $SL_2(\mathbf{R})$ such that Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, and $J(C)$ a complex number defined for each class C of elements of $\Gamma\alpha\Gamma$ under a certain equivalence. The double cosets $\Gamma\alpha\Gamma$ and $\Gamma\alpha^{-1}\Gamma$ act on A_m and B_{2-m} respectively, under some conditions. An *integral form* of weight m is a holomorphic function $f(z)$ on $\mathfrak{H}$ which satisfies $f(\gamma(z))/f(z) = t(\gamma)(d\gamma(z)/dz)^{-m/2}$ for every $\gamma \in \Gamma$ with a certain constant factor $t(\gamma)$, and which is holomorphic at every cusp; an integral form is called a *cusp form* if it vanishes at every cusp.

If m is an integer >2, then $B_{2-m}=\{0\}$. The formula in this case was obtained by Selberg [5] and Eichler [2]. If $m=2$, B_{2-m} consists of the constants, and therefore $\mathrm{tr}\,(\Gamma\alpha^{-1}\Gamma\,|\,B_{2-m})$ is simply the number of right or left cosets in $\Gamma\alpha^{-1}\Gamma$. This case is also included in [2]. It should also be mentioned that the generalized Riemann-Roch theorem of Weil [8] is closely related to the above formula when α belongs to the normalizer of Γ.

Although our formula is given for an arbitrary rational m, the cases of integral and half integral weight with respect to an arithmetic Γ seem most significant. If m is a half integer >2, we have again $B_{2-m}=\{0\}$, and the formula is of the same nature as in the case of integral $m>2$. However, if $m=3/2$, both A_m and B_{2-m} can be non-trivial. Especially if Γ is a congruence subgroup of $SL_2(\mathbf{Z})$, it is conjecturable that $B_{\frac{1}{2}}$ is spanned by theta series of the type

$$\Sigma_n \psi(n) \exp(2\pi i n^2 r z)$$