LIMIT POINTS OF KLEINIAN GROUPS AND FINITE SIDED FUNDAMENTAL POLYHEDRA

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Let G be a discrete subgroup of $SL(2, C)/\{\pm 1\}$. Then G operates as a discontinuous group of isometries on hyperbolic 3-space, which we regard as the open unit ball $\mathbf{B}^3$ in Euclidean 3-space $\mathbf{E}^3$. G operates on $\mathbf{S}^2$, the boundary of $\mathbf{B}^3$, as a group of conformal homeomorphisms, but it need not be discontinuous there. The set of points of $\mathbf{S}^2$ at which G does not act discontinuously is the *limit set* $\Lambda(G)$.

If we fix a point 0 in $\mathbf{B}^3$, then the orbit of 0 under G accumulates precisely at $\Lambda(G)$. The approximation is, however, not uniform. We distinguish a class of limit points, called *points of aproximation*, which are approximated very well by translates of 0. The set of points of approximation includes all loxodromic (including hyperbolic) fixed points, and includes no parabolic fixed points. In § 1 we give several equivalent definitions of point of approximation, and derive some properties. We remark that these points were first discussed by Hedlund [7].

Starting with a suitable point 0 in $\mathbf{B}^3$, we can construct the Dirichlet fundamental polyhedron P_0 for G. It was shown by Greenberg [5] that even if G is finitely generated, P_0 need not have finitely many sides. Our next main result, given in § 2, is that if P_0 is finite-sided, then every point of $\Lambda(G)$ is either a point of approximation or a cusped parabolic fixed point (roughly speaking a parabolic fixed point is cusped if it represents the right number of punctures in $(\mathbf{S}^2 - \Lambda(G))/G$).

The above theorem has several applications: one of these is a new proof of the following theorem of Ahlfors [1].

If P_0 has finitely many sides, then the Euclidean measure of $\Lambda(G)$ is either 0 *or* 4π.

Our next main result, given in § 3, is that the above necessary condition for P_0 to have finitely many sides is also sufficient. In fact, we prove that any convex fundamental polyhedron G has finitely many sides if and only if $\Lambda(G)$ consists entirely of points of