PARTIAL REGULARITY OF MAPPINGS BETWEEN EUCLIDEAN SPACES

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1. Introduction

Let f be a locally bounded function from a p-dimensional Euclidean space E_p to a q-dimensional Euclidean space F_q. For a given subset Λ of $E_p \times F_q$ we will consider conditions on f of the following type: for each $(\xi, \eta) \in \Lambda$, $\xi \in E_p$, $\eta \in F_q$, the function $x \to \langle \eta, f(x) \rangle$ has a certain regularity property in the direction ξ. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in F_q. The problem is to determine the condition on Λ in order that these conditions on f imply a corresponding (unrestricted) regularity property for the function f.

The answer to these problems is formulated in terms of the following two algebraic conditions on Λ. Let $\mathbf{R}$ denote the real numbers.

(A) *if* Φ *is a bilinear form* $(E_p, F_q) \to \mathbf{R}$ *and* $\Phi(\Lambda) = 0$, *then* $\Phi = 0$.

$(\hat{A})$ *if* Φ *is a bilinear form* $(E_p, F_q) \to \mathbf{R}$ *of rank* 1 *and* $\Phi(\Lambda) = 0$, *then* $\Phi = 0$.

As examples of our results we mention the following. If the regularity property is continuity or infinite differentiability, the condition $(\hat{A})$ is necessary and sufficient for an assertion of the above-mentioned type to hold. If we consider continuity of the first derivatives, the condition (A) plays the same role. If f is locally bounded and $\langle \eta, f \rangle$ is constant in the direction ξ for each $(\xi, \eta) \in \Lambda$, then it follows that f is constant if and only if (A) holds. The same assumption implies that f is a polynomial, if and only if $(\hat{A})$ holds.

If (A) holds, Λ contains at least pq elements. On the other hand, there exist subsets Λ of $E_p \times F_q$, which satisfy $(\hat{A})$ and contain only $p+q-1$ elements. If $q=1$, then (A) and $(\hat{A})$ are equivalent and mean simply that the linear hull of $\{\xi; (\xi, \eta) \in \Lambda, \eta \neq 0\}$ is equal to E_p. An analogous statement holds of course if $p=1$. Our results are trivial in case p or q is equal to one.

The above-mentioned problem becomes particularly interesting if the regularity in question is described by the modulus of continuity. Then both of the conditions (A) and