SUPPORTS AND SINGULAR SUPPORTS OF CONVOLUTIONS

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1. Introduction

If f is a distribution in R^n we write supp f (resp. sing supp f) for the smallest closed set outside which $f=0$ (resp. $f \in C^\infty$). Then the convolution theorem of Titchmarch [13], extended from one to n dimensions by Lions [10], states that

$$\text{ch supp } (f_1 * f_2) = \text{ ch supp } f_1 + \text{ ch supp } f_2; \quad f_1, f_2 \in \mathcal{E}'. \tag{1.1}$$

Here we have used the notation ch A for the convex hull of a set A in R^n and written

$$A+B=\{x+y;\ x \in A,\ y \in B\}$$

if A and B are subsets of R^n; below $A-B$ will be defined similarly.

The aim of this paper is to prove results similar to (1.1) where supports are replaced by singular supports. In Hörmander [5] it was proved in perfect analogy with (1.1) that

$$\text{ch sing supp } (f_1 * f_2) = \text{ ch sing supp } f_1 + \text{ ch sing supp } f_2 \tag{1.2}$$

provided that $f_1, f_2 \in \mathcal{E}'$ and either supp f_1 or supp f_2 consists of a finite number of points, a result due to F. John and B. Malgrange when the number of points is one. When f_2 is hypoelliptic in the sense of Ehrenpreis [4] it was also proved in Hörmander [6] that

$$\text{ch sing supp } f_1 \subset \text{ch sing supp } (f_1 * f_2) - \text{ ch sing supp } f_2, \tag{1.3}$$

which is a weakened form of the non-trivial part of (1.2) that the left-hand side of (1.2) contains the right-hand side. However, not even this weaker result can be valid for arbitrary f_2, for it may happen that $f_1 * f_2 \in C_0^\infty$ although neither f_1 nor f_2 is in C_0^∞. In fact, Ehrenpreis [4] has proved that every $f_1 \in \mathcal{E}'$ with $f_1 * f_2 \in C_0^\infty$ belongs to C_0^∞ if and only if the Fourier transform $\hat{f}_2$ of the distribution $f_2 \in \mathcal{E}'$ is slowly decreasing in the sense that for some constant A