SOME GEOMETRIC AND ANALYTIC PROPERTIES OF HOMOGENEOUS COMPLEX MANIFOLDS

Part II: Deformation and Bundle Theory

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9. Deformation Theory; Part I

(i) The Infinitesimal Theory

Let Y be a compact complex manifold and suppose that on the differentiable manifold Y_d we are given a 1-parameter family of complex structures $Y_t(Y_0 = Y)$. If $U = \{U_i\}$ is a covering of Y by complex coordinate neighborhoods, with coordinates $(z_i^1, \ldots, z_i^n)$ in U_i, the structure on Y_t is given by transition functions

$$z_j^r(z_i, t) = f_{ij}^r(z_i^1, \ldots, z_i^n; t);$$

letting

$$\theta_{ij}^r = \left[\frac{df_{ij}^r}{dt}\right]_{t=0}$$

and

$$\theta_{ij} = (\theta_{ij}^1, \ldots, \theta_{ij}^n),$$

the deformation Y_t of Y is represented infinitesimally (or linearly) by the 1-cocycle $\theta_{ij} \in H^1(N(U), \Theta)$ ($N(U)$ = nerve of U). Further details concerning the relation between the variation of structure of Y and its parametrization by $H^1(Y, \Theta)$ are given in [19] and [20]; we shall be concerned with the special cases when Y = non-Kähler C-space or $Y = \hat{X} \times T^{2a}$ where $\hat{X}$ is a Kähler C-space. We remark that by corollary 1 to Theorem 2, the structure of $\hat{X}$ is infinitesimally rigid. By way of notation, we let $X = G/U$ be a non-Kähler C-space with fundamental fibering $T^{2a} \to X \to \hat{X}$, $\hat{X} = G/\hat{U}$ a Kähler C-space, and we set $\hat{X}^b = \hat{X} \times T^{2b}$; the manifolds $\hat{X}^b$ are the most general compact homogeneous Kähler manifolds. If $Y = X$ or $\hat{X}^b$ (where b may be zero), the group $H^1(Y, \Theta_Y)$ is a representation space and for us this interpretation will be crucial.