Bounded point evaluations and balayage

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0. Introduction

The problem treated in this paper can be formulated as follows: Define W as the closure of $C_0^\infty(\mathbf{R}^d)$ in the norm $\{\int |\operatorname{grad} f|^2 dx\}^{1/2}$. If E is open and bounded let $H(E)$ be the subspace of W consisting of functions harmonic in E. If E is compact let $H(E)$ be the closure in W of functions harmonic in some neighbourhood of E. (Here $d \geqq 3$; if $d=2$ we replace $C_0^\infty(\mathbf{R}^d)$ by $C_0^\infty(D)$ — D the open unit disc — and we assume that $E \subset\subset D$.)

A point $a \in \bar{E}$ for which the mapping $f \to f(a)$ is bounded on $H(E)$ is called a *bounded point evaluation* (BPE) for $H(E)$ and our aim is to characterize these points.

In Fernström—Polking [4] a similar problem is treated for a more general elliptic differential operator acting on $L^p(E)$, E a compact set in $\mathbf{R}^d$. (For a more detailed discussion on BPEs we refer to that paper and the references there.) Compared to [4] we are here dealing with a special case; we can then make use of other methods, specific to this problem. In particular we apply the operation of sweeping out a measure, balayage. We can also take care of the case when E is an open set.

We get several conditions characterizing the BPEs. Two of these are to be stressed cf. [4, theorems 1 and 3]): The first one is that the fundamental solution of the Laplace operator with a pole at a (the function $x \to \text{const}\, |x-a|^{2-d}$ if $d \geqq 3$) can be continued from E^c to $\mathbf{R}^d$ so as to be an element of $H(E)$. Moreover, this new function is the Newton potential of the Dirac measure δ_a at a swept out onto E^c. The second one is a Wiener type condition, the BPEs are precisely the points for which E^c is subject to a certain kind of thinness.

The main references are Cartan [2] and Landkof [6]. Also Hedberg [5] is a good reference for sections I and II.

We will use the following notation:

For a (Borel) set $B \subset \mathbf{R}^d$ ($d \geqq 3$ in sections I—IV, $d=2$ in section V) B^0 its interior, $\bar{B}$ its closure, ∂B its boundary and B^c its complement. $C_0^\infty(G)$ is the space