Polynomially convex hulls and analyticity

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Introduction

We denote by z, w the coordinates in $\mathbf{C}^2$ and we write π for the projection which sends $(z, w) \to z$. Let Y be a compact subset of $\mathbf{C}^2$ with $\pi(Y)$ contained in the unit circle. We denote by $\hat{Y}$ the polynomially convex hull of Y. For λ in $\mathbf{C}$ we put

$$\pi^{-1}(\lambda) = \{(z, w) \in \hat{Y} \mid \pi(z, w) = \lambda\}.$$

We assume that $\pi^{-1}(\lambda) \neq \emptyset$ for some λ with $|\lambda| < 1$. Then $\pi^{-1}(\lambda) \neq \emptyset$ for each λ in the open unit disk.

Under various conditions $\hat{Y} \setminus Y$ has been shown to possess analytic structure. In particular we have ([4], [5]):

Theorem. *If $\pi^{-1}(\lambda)$ is finite or countably infinite for each λ in $|\lambda| < 1$, then $\hat{Y} \setminus Y$ contains an analytic variety of dimension* 1.

The object of this note is to show that no such conclusion holds in general.

Theorem 1. *There exists a compact subset Y of $\mathbf{C}^2$ with $\pi(Y) \subseteq \{|z| = 1\}$ such that $\pi(\hat{Y}) = \{|z| \leqq 1\}$ and $\hat{Y} \setminus Y$ contains no analytic variety of positive dimension.*

Our construction proceeds by modifying the idea which was used by Brian Cole in [1] (see also [3], Theorem 20.1) to prove the infinite-dimensional analogue of Theorem 1.

In the famous example of a hull without analytic structure given by Stolzenberg in [2] the set whose hull is taken and the hull have the same coordinate projections. In our example the projection $\pi(Y)$ is a proper subset of the projection $\pi(\hat{Y})$.

Note. By a change of variable we may replace the unit circle and unit disk by the circle $|z| = 1/2$ and the disk $|z| \leqq 1/2$, and we shall prove Theorem 1 for this case. The convenience that results is that for $|a|$, $|b| \leqq 1/2$, $|a-b| \leqq 1$.