On the propagation of singularities for pseudo-differential operators of principal type

Nils Dencker

1. Introduction

Let P be a properly supported pseudo-differential operator of order m on a C^∞ manifold X. We shall assume that the symbol of P is a sum of terms homogeneous of degree $m, m-1, \ldots$ and we denote the principal symbol by p.

Definition 1.1. P is said to satisfy condition (P) if there is no C^∞ complex valued function q in $T^*X\setminus 0$ such that $\operatorname{Im} qp$ takes both positive and negative values on a bicharacteristic of $\operatorname{Re} qp$ where $q \neq 0$.

By a bicharacteristic of $\operatorname{Re} qp$ we mean an integral curve of the Hamilton field $\operatorname{Re} H_{qp}$ on which $\operatorname{Re} qp$ vanishes. (Some authors call this a null-bicharacteristic.) We say that P is of principal type if $dp \neq 0$ when $p=0$. For operators of principal type satisfying condition (P) and with no bicharacteristics trapped over a point, Nirenberg and Treves [5] proved local solvability when the principal symbol is analytic. Beals and Fefferman [1] extended their result to the C^∞ case. Hörmander [3] proved semi-global solvability by studying the propagation of singularities for the adjoint operator. In this paper we shall study the case which was left open in [3].

Definition 1.2. We denote by $\mathscr{C}_3$ the set of $(x, \xi) \in T^*X\setminus 0$ such that $p(x, \xi)=0$ and $\operatorname{Im} qp$ vanishes of third order at (x, ξ) for some $q \in C^\infty(T^*X\setminus 0)$ such that $q(x, \xi) \neq 0$.

Observe that $\mathscr{C}_3$ contains the set $\mathscr{C}_{13}$ defined by Hörmander [3], for which there are also global conditions. The definition implies that a bicharacteristic γ of, say, $\operatorname{Re} p$ is a one dimensional bicharacteristic of p as long as it remains in $\mathscr{C}_3$, that is, $p=0$ on γ and $H_p \neq 0$ is proportional to the tangent vector.

When studying the singularities we shall use the Sobolev spaces $H_{(s)}$ of distributions which are mapped into L^2 by any pseudo-differential operator of order s.