On John and Nirenberg's theorem

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Introduction

A well-known theorem by John and Nirenberg states that for a function f in $BMO(\mathbf{R}^n)$ with $\|f\|_{BMO}=K$ we have for every cube Q with sides parallel to the axes:

$$(1) \qquad \left|\{x\in Q;\ |f(x)-a_Q| > \sigma\}\right| \leqq c_1 e^{-c_2\sigma K^{-1}}|Q|.$$

The constant c_2 which is obtained normally is of the form 2^{-cn}. In the paper [2] John and Nirenberg claim that the constant c_2 can be improved to be of the order $\log n/n$. (c_1 is an absolute constant e.g. 2.)

In this paper we introduce the more general notion of *a false cube* and an associated BMO-norm, $\|f\|'_{BMO}$. We will show that (1) is true with this norm for all false cubes Q with a constant c_2 which then is *independent of* n (Theorem 1). We also will show (Theorem 2) that the quotient of $\|f\|'_{BMO}$ and $\|f\|_{BMO}$ is at most of the order $\sqrt{n}$, which means that we can improve c_2 in (1) to the order of $n^{-1/2}$ and at the same time allow Q to be any false cube.

Definitions and notations

A *cube* will always mean a cube in $\mathbf{R}^n$ with sides parallel to the axes.

A *false cube* is an n-dimensional rectangle in $\mathbf{R}^n$ whose sides are parallel to the axes and for some s have side lengths either s or $2s$, i.e. its proportions are $2\times2\times\ldots\times2\times1\times1\times\ldots\times1$.

The *Lebesgue measure* of a set E is denoted by $|E|$. If f is a real-valued function in L^1_{loc} we define the *sharp function*, $f^\#$, by

$$(2) \qquad f^\#(t) = \sup_{Q\ni t}\frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dx\,dy,$$

where the supremum is taken over all cubes Q containing t.