Approximation of plurisubharmonic functions

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Introduction

Let $PSH(X)$, $C(X)$, $C^\infty(X)$ denote respectively the plurisubharmonic functions, the continuous functions and the smooth functions defined on a neighborhood of a set $X \subset \mathbf{C}^n$. Let Ω be a domain in $\mathbf{C}^n$. How are $PSH(\Omega)$, $PSH(\Omega)\cap C(\Omega)$, $PSH(\Omega)\cap C(\bar{\Omega})$ and $PSH(\bar{\Omega})\cap C(\bar{\Omega})$ related? What if we replace $C(\Omega)$ or $C(\bar{\Omega})$ by $C^\infty(\Omega)$, $C^\infty(\bar{\Omega})$. More specifically, is it possible to approximate elements of one of these classes with elements of a smaller one.

Richberg [4] showed that for every strictly plurisubharmonic $f \in C(\Omega)$ and every $\varepsilon(z) \in C(\Omega)$, $\varepsilon(z) > 0$, there exists a strictly plurisubharmonic $\Phi \in C^\infty(\Omega)$ such that $0 < \Phi(z) - f(z) < \varepsilon(z)$. The first author, [2], exhibited a smooth Hartogs domain D in $\mathbf{C}^2$ and a plurisubharmonic function f on it, so that f cannot be approximated from above with functions in $PSH(D)\cap C(D)$. Sibony [5] showed that if Ω is a pseudoconvex domain with C^∞-boundary, then $f \in PSH(\Omega)\cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ with $\Phi \in PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$. He asked if this were true for pseudoconvex domains with C^1 boundary also.

Section 1 deals with Sibony's question. In Theorem 1 it is answered positively for *arbitrary* bounded domains with C^1-boundary. Our proof is entirely different from Sibony's. Assuming in addition that Ω is pseudoconvex, we also show that every $f \in PSH(\Omega)\cap C(\Omega)$ can be approximated uniformly on compact sets with $\Phi \in PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$, Theorem 2. Section 2 contains an example that Theorem 2 without the pseudoconvexity assumption is false. In Section 3 and 4 we show that if Ω is a Reinhardt or a tube domain in $\mathbf{C}^n$, then $f \in PSH(\Omega)$ is pointwise the limit of a monotonically decreasing sequence $\Phi_j \in PSH(\Omega)\cap C^\infty(\Omega)$.

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