A new proof of a Paley—Wiener type theorem for the Jacobi transform

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1. Introduction

Jacobi functions $\varphi_\lambda(t)$ of order (α, β) are the eigenfunctions of the differential operator $(\Delta(t))^{-1}(d/dt)(\Delta(t)\, d/dt)$, $\Delta(t)=(e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}$, such that $\varphi_\lambda(0)=1$, $\varphi_\lambda'(0)=0$. The Jacobi transform

$$f^\wedge(\lambda) = \left(2^{1/2}/\Gamma(\alpha+1)\right)\int_0^\infty f(t)\,\varphi_\lambda(t)\,\Delta(t)\,dt, \tag{1.1}$$

which generalizes the Mehler-Fok transform, was studied by Titchmarsh [23, §4. 17], Olevskiĭ [21], Braaksma and Meulenbeld [2], Flensted—Jensen [9], [11, §2 and §12] and Flensted—Jensen and Koornwinder [12]. Some papers by Chébli [3], [4], [5] deal with a larger class of integral transforms which includes the Jacobi transform. An even more general class was considered by Braaksma and De Snoo [24].

In the present paper short proofs will be given of a Paley—Wiener type theorem and the inversion formula for the Jacobi transform. The L^2-theory, i.e. the Plancherel theorem, is then an easy consequence. These results were earlier obtained by Flensted—Jensen [9], [11, §2] and by Chébli [5]. However, to prove the Paley—Wiener theorem these two authors needed the L^2-theory, which can be obtained as a corollary of the Weyl—Stone—Titchmarsh—Kodaira theorem about the spectral decomposition of a singular Sturm—Liouville operator (cf. for instance Dunford and Schwartz [6, Chap. 13, §5]). The proofs presented here exploit the properties of Jacobi functions as hypergeometric functions and no general theorem needs to be invoked. Furthermore, it turns out that the Paley—Wiener theorem, which was proved by Flensted—Jensen [11, §2] for real $\alpha, \beta, \alpha>-1$, holds for all complex values of α and β.

The key formula in this paper is a generalized Mehler formula

$$(\Gamma(\alpha+1))^{-1}\Delta(t)\,\varphi_\lambda(t) = \pi^{-1/2}\int_0^t \cos\lambda s\, A(s,t)\,ds, \tag{1.2}$$