L^p estimates for strongly singular convolution operators in $\mathbf{R}^n$

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Let the operator T_K be defined by $T_K f = K * f$, $f \in C_0^\infty(\mathbf{R}^n)$, where K is a distribution with compact support in $\mathbf{R}^n$, locally integrable outside the origin and satisfying the conditions

$$A(\alpha) \qquad |\hat{K}(\xi)| \leqq B(1+|\xi|)^{-n\alpha/2}, \quad \xi \in \mathbf{R}^n,$$

and

$$B(\theta) \qquad \int_{|x|>2|y|^{1-\theta}} |K(x-y)-K(x)|\, dx \leqq B, \quad |y| < b.$$

Here $\hat{K}$ is the Fourier transform of K, B and b denote positive constants and $0 \leqq \alpha \leqq \leqq \theta < 1$.

The conditions $A(0)$ and $B(0)$ are satisfied by the well-known Calderón—Zygmund kernels and in the case $\alpha = \theta = 0$ it is known that T_K can be extended to a bounded linear operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

In the case $\alpha > 0$ an example of a kernel satisfying the above conditions can be obtained in the following way. Let L be defined by setting

$$\hat{L}(\xi) = \psi(\xi) e^{i|\xi|^a} |\xi|^{-n\alpha/2}, \quad \xi \in \mathbf{R}^n,$$

where $a = (n\alpha(1-\theta)+2\theta)/(n(1-\theta)+2)$ and ψ is a C^∞ function which vanishes near the origin and is equal to one for $|\xi|$ large. Then $L = K + M$, where K satisfies the above conditions and M is an L^1 function with $\hat{M}(\xi) = O(|\xi|^{-N})$, $|\xi| \to \infty$, for all N. In fact it was proved by S. Wainger [7] that $K(x)$ is essentially equal to $c_1 |x|^{-n-\lambda} e^{ic_2|x|^{a'}}$ close to the origin, where $\lambda = n(a-\alpha)/2(1-a)$ and $1/a + 1/a' = 1$. Hence $|\text{grad } K(x)| \leqq \leqq C|x|^{-n-\lambda-1+a'}$, from which it follows that $B(\theta)$ is satisfied. The L^p theory for operators obtained by convolution with kernels of this type has been studied by I. I. Hirschmann [4] and Wainger [7].