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On local integrability of fundamental solutions

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1. Introduction

Let $P(D)$, $D=-i\partial/\partial x$, be a partial differential operator in $\mathbf{R}^n$ with constant coefficients. In my thesis [1] I proved that $P(D)$ is hypoelliptic if and only if one of the following equivalent conditions is fulfilled:

(i) $\operatorname{Im}\zeta\to\infty$ if $\mathbf{C}^n\ni\zeta\to\infty$ and $P(\zeta)=0$;

(ii) $P(\xi)\neq0$ for large $\xi\in\mathbf{R}^n$, and $P^{(\alpha)}(\xi)/P(\xi)\to0$ when $\mathbf{R}^n\ni\xi\to\infty$, if $\alpha\neq0$.

The sufficiency was proved by constructing a fundamental solution, that is, a *distribution* E with $P(D)E=\delta$, and verifying that (ii) implies that $E\in C^\infty$ in $\mathbf{R}^n\setminus\{0\}$. In a conversation with Marcel Riesz, who had been my mentor but was then retired, he reproached me for relying on the notion of distribution and told me that I ought to prove that E is in fact a locally integrable function. This reaction was quite typical of the reluctance of the mathematical community to accept the notion of distribution. It was not unexpected, and I had in fact avoided using distributions as far as I could.

Although it is quite irrelevant for the purposes of [1], I have never quite been able to dismiss the question whether the fundamental solutions of a hypoelliptic operator in $\mathbf{R}^n$ are always locally integrable. In Section 2 we shall prove that the answer is positive when $n=2$, but in Section 4 we shall give an example proving that the answer is negative for every $n\geq14$. At last this settles the question except for dimensions $3,\dots,13$, and proves that distributions are essential and not only convenient in this context.

If $P(D)$ is an elliptic differential operator then $P^{(\alpha)}(D)E$ is essentially the inverse Fourier transform of $P^{(\alpha)}(\xi)/P(\xi)$, which behaves at infinity as a function which is homogeneous of degree $-|\alpha|$. When $|\alpha|=1$ it follows that $P^{(\alpha)}(D)E$ is singular at the origin as a homogeneous function of degree $1-n$, which gives that $P^{(\alpha)}(D)\in L^p_{\text{loc}}$ if and only if $p<n/(n-1)$. For arbitrary $\alpha\neq0$ we have $P^{(\alpha)}(D)E\in L^p_{\text{loc}}$ if $1/p>1-|\alpha|/n$. More generally, if $P(D)$ is semielliptic in the sense of [2, Chapter XI, p. 67] of orders $m_1\geq m_2\geq\dots\geq m_n$, then it is easy to see that $P^{(\alpha)}(D)E\in L^p_{\text{loc}}$