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Best approximation in the supremum norm by analytic and harmonic functions

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1. Introduction

In this paper, we study the problem of finding, for a given bounded measurable function f on a domain Ω in $\mathbf{R}^n$, a harmonic function on Ω that best approximates f in the supremum norm, as well as (when $n=2$) the corresponding problem of approximating f by analytic functions. The analogous problem of approximating a bounded measurable function on the boundary of a plane domain (especially, the unit disk) by the *boundary values* of bounded analytic functions in the interior has been studied very extensively (see, e.g. [G]), but the present problem (which, as we shall see, is quite different in character) has received very little attention. There have been some studies, by Luecking [L1], [L2], Hintzman [H1], [H2] and Romanova [R1], [R2], pertaining to approximation by analytic functions.

Concerning the harmonic approximation problem in $\mathbf{R}^n$, its study seems to originate in a paper of Hayman, Kershaw and Lyons [HKL] from 1984. Our main motivation has been to refine and extend some of the results of that paper. This we have been able to do, in part by making greater use of functional analysis than they do. Those tools, *per se*, are well known. In the interest of a unified presentation, we have included proofs of some previously known results. The main novelty of the paper is Theorem 4.1 which, in the two-dimensional case, gives an affirmative answer to a question asked by Walter Hayman in 1984. Theorem 3.5 contains an answer to another question of Hayman. (These questions were contributed to the session on "New and Unsolved Problems" at the conference where [HKL] was presented, see p. 608 of the conference proceedings.) Also, several of our counterexamples are new, or sharper than previously known ones of similar character.

The present paper can be seen as complementary to [KMS], where the analogous problems were studied in L^p norm (with respect to volume measure) for $1\leq p<\infty$.