Partial regularity for minima of variational integrals

Mariano Giaquinta and Per–Anders Ivert*)

The purpose of this paper is to study regularity properties of vector-valued functions minimizing variational integrals of the form

$$\mathscr{F}(u) = \int_\Omega F(x, u(x), Du(x))\, dx$$

where Ω is a domain in $\mathbf{R}^n$ and $F(x, u, p)$ is a continuous function, convex in p and growing, for $|p|$ large, like $|p|^m$, $m \geqq 2$.

We will derive partial regularity, i.e. continuity except on a closed set of measure zero, for the derivatives of the minima of $\mathscr{F}$ under the assumption that F is twice continuously differentiable in p but only Hölder continuous in x and u, which means that the functional $\mathscr{F}$ is in general non-differentiable. This extends previous results of Giaquinta—Giusti [5] and Ivert [7], where the case $m=2$ is treated.

Although the techniques employed are much in the same spirit as the ones used in [5] and [7], the additional difficulties which arise for $m>2$ require some technical adjustments which may be of some independent interest.

Let us state our assumptions precisely:

General assumptions. *Let Ω be a domain in euclidean n-space $\mathbf{R}^n$, $n \geqq 3$, let N be a positive integer and let $F\colon \Omega \times \mathbf{R}^N \times \mathbf{R}^{Nn}$ be a function satisfying for all $x, y \in \Omega$, $u, v \in \mathbf{R}^N$ and $p, q \in \mathbf{R}^{Nn}$:*

(i) $|p|^m \leqq F(x, u, p) \leqq c_0(1+|p|^2)^{m/2}$

(ii) $|F(x, u, p) - F(y, v, p)| \leqq c_0(1+|p|^2)^{m/2}(|x-y|^\sigma + |u-v|^\sigma)$

(iii) $|F_p(x, u, p)| \leqq c_0(1+|p|^2)^{(m-1)/2}$

(iv) $c_0^{-1}(1+|p|^2)^{(m-2)/2}|q|^2 \leqq F_{p_\alpha^i p_\beta^j}(x, u, p)\, q_\alpha^i q_\beta^j \leqq c_0(1+|p|^2)^{(m-2)/2}|q|^2$

(v) $|F_{pp}(x, u, p) - F_{pp}(x, u, q)| \leqq (1+|p|^2+|q|^2)^{(m-2)/2}\,\omega(|p-q|^2)$.

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