Sharp estimates of uniform harmonic majorants in the plane

Matts Essén

1. Introduction

If f is an analytic function in the unit disk U, the Dirichlet integral $D(f)$ is defined by

$$D(f) = \left(\int_U |f'(z)|^2 \, dx\, dy/\pi\right)^{1/2}, \quad z = x+iy.$$

The following result is due to A. Chang and D. Marshall (cf. [4], [7]). It is inspired by work of A. Beurling and J. Moser (cf. [3], [8]).

Theorem A. *There is a constant* $C<\infty$ *such that if f is analytic in* U, $f(0)=0$ *and* $D(f)\leqq 1$, *then*

$$\int_0^{2\pi} \exp\left(|f(e^{i\theta})|^2\right) d\theta \leqq C.$$

If f is univalent, $\pi D(f)^2$ is the area $|f(U)|$ of the range $f(U)$ of f. What can be said about functions f which are not necessarily univalent if the assumption on f is replaced by $|f(U)|\leqq\pi$? Can this condition on the area of $f(U)$ be generalized?

Let D be an open, connected subset in the plane and let $\theta(r)=\left|\{\theta\colon re^{i\theta}\in D\}\right|$. Let F be the class of locally bounded functions $\Psi\colon(0,\infty)\to(0,\infty)$ which have the following properties:

i) near the origin, we have $\Psi(r)=cr^2$, c constant,
ii) for each $a>0$, we have $\inf_{r\geqq a}\Psi(r)>0$,
iii) there exists $R>0$ such that Ψ is increasing on (R,∞); in the interval $(0,R)$, Ψ is continuous except possibly at finitely many points.

Let $p(r)=\int_0^r (\Psi(t))^{1/2}dt/t$ and let $\Phi(r)=\exp\left(p(r)^2\right)$. The function $\Phi(|z|)$ is subharmonic in $\{|z|>R\}$: this is clear since $r\Phi'(r)$ is increasing for $r>R$ and $\Delta\Phi=r^{-1}(d/dr)\left(r\Phi'(r)\right)$. Natural examples of functions satisfying these conditions are given in Corollaries 2 and 3.