Non commutative Khintchine and Paley inequalities

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Introduction

Let (ε_n) be a sequence of independent ± 1-valued random variables on a probability space (D, μ) with $\mu(\varepsilon_n=1)=\mu(\varepsilon_n=-1)=1/2$ (for instance the Rademacher functions on the Lebesgue interval).

Let X be a Banach space and let (x_n) be a sequence in X. In recent years, a great deal of work has been devoted to try to find "explicit" necessary and sufficient conditions for the series

$$\sum_{n=1}^{\infty} \varepsilon_n x_n \tag{0.1}$$

to converge (in norm) almost surely, see for instance [Ka], [MP] and [LeT].

Equivalently the problem reduces to find an "explicit expression" equivalent to the norm defined by

$$\|(x_n)\| = \left(\int \left\|\sum \varepsilon_n x_n\right\|^2 d\mu\right)^{1/2} \tag{0.2}$$

considered as a norm on the set of all finitely supported sequences (x_n) in X. While a satisfactory solution seems hopeless at the moment for an *arbitrary* space X, there are cases for which the answer is known to be very simple and as complete as possible. For instance, if X is the Banach space $L_p(\Omega, \sum, m)$ $(1 \leqq p < \infty)$ the classical Khintchine inequalities (cf. [LT, I.d.6]) and Fubini's theorem imply that there is an absolute constant C such that, for all x_n in $X=L_p(\Omega, \sum, m)$, we have

$$\frac{1}{C}\|(x_n)\| \leqq \left\|\left(\sum |x_n|^2\right)^{1/2}\right\|_X \leqq C\|(x_n)\|. \tag{0.3}$$

This solves the above mentioned problem when $X=L_p(m)$. More generally, as shown by Maurey (cf. [LT] p. 50) (0.3) remains valid when X is a Banach lattice iff X is q-concave for some $q<\infty$. In this paper, we investigate what remains of (0.3) when X is a *non-commutative* L_p-space (or a non commutative analogue of a Banach