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Smoothness of the boundary function of a holomorphic function of bounded type

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In a recent paper [5] the author has proved the following theorem:

Theorem A. *Let $f(z)$ be holomorphic and of bounded type in $|z|<1$ and suppose its radial boundary values $f(e^{i\theta})$ coincide almost everywhere with a function $F(\theta)$ of period 2π and class C^∞ such that for some positive A,*

$$\max_{\theta} |F^{(n)}(\theta)| \leqslant (An)^{2n}, \quad n=1, 2, \dots . \tag{1}$$

Then f is bounded in $|z|<1$, and consequently f and all its derivatives are uniformly continuous in $|z|<1$. If the right side of (1) were replaced by $(An)^{pn}$ for some $p>2$, the resulting theorem would be false.

In [5] this theorem was proved by a method based on weighted polynomial approximation. It was also conjectured that theorem A is true "locally", that is, if in (1) the max is over the interval $[\theta_1, \theta_2]$ instead of $[0, 2\pi]$, a corresponding conclusion holds for the neighbourhood of the arc joining $e^{i\theta_1}$ and $e^{i\theta_2}$.

The main purpose of the present paper is to prove this conjecture. An altogether different, more direct, method is employed. Actually the relevant theorem (Theorem 1 below) is formulated for a half plane rather than a disk—the version for a disk (or indeed any domain having an analytic arc on its boundary) can easily be deduced from that for a half plane.

We also take this opportunity to settle another point from [5]. On p. 334 we stated that "it seems most plausible" an analogous theorem holds for *meromorphic* functions of bounded type, if (1) is replaced by the stronger condition that F belong to a Denjoy–Carleman quasi-analytic class (that is, under such a hypothesis on F, f is bounded, and in particular is free of poles, near the boundary). This "plausible" assertion is false, as shown in the corollary to Theorem 2 below. In fact, no majorant short of $(An)^n$, which implies analytic continuation of f across the boundary arc in question, can guarantee the absence of poles near the boundary. (But, for a plausible conjecture, see the concluding remarks.)

Before turning to Theorem 1, we wish to recall some facts concerning functions of bounded type; we formulate them for functions in a disk, the definitions and results for functions in a half plane are similar. For the notions of "inner" and "outer" function see Hoffman [2] (also [4] but in [4] these terms are not used). A *singular function* is an inner function without zeros. A complete divisibility theory exists for singular functions (see [2] pp. 84–85); and every holomorphic function f