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Compression semigroups of open orbits in complex manifolds

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Introduction

Let $G_{\mathbf{C}}$ be a connected complex Lie group and $G \subseteq G_{\mathbf{C}}$ a real form, i.e., there exists an antiholomorphic involution σ of $G_{\mathbf{C}}$ such that $G = G_{\mathbf{C}}^{\sigma} = \{g \in G_{\mathbf{C}} : \sigma(g) = g\}$ is the group of fixed points. Now let $M = G_{\mathbf{C}}/P$ be a complex homogeneous space and suppose that the G-orbit $\mathcal{O}$ of the base point is open. We are interested in the semigroup

$$S(P) := \{g \in G_{\mathbf{C}} : gGP \subseteq GP\} = \{g \in G_{\mathbf{C}} : g.\mathcal{O} \subseteq \mathcal{O}\}$$

of *compressions* of the open G-orbit.

Such semigroups play a central role in the theory holomorphic extensions of unitary representations of the group G (cf. [HOØ], [FHO], [Ols], [N5], [N6], [N7], [St]).

More concretely we are dealing with the following two classes of homogeneous spaces, namely with complex flag manifolds and with certain embeddings of complex coadjoint orbits into complex homogeneous spaces.

The main results for complex flag manifolds are fairly easy to describe. Since everything decomposes nicely according to the decomposition of $G_{\mathbf{C}}$ into simple factors, we may assume that $G_{\mathbf{C}}$ is simple. In this case three mutually exclusive possibilities occur (cf. Proposition II.3, Theorem III.14):

(1) $S(P) = G_{\mathbf{C}}$.

(2) $\operatorname{int} S(P) = \emptyset$.

(3) $G_{\mathbf{C}} \neq \operatorname{int} S(P) \neq \emptyset$. Then G is a Hermitean simple Lie algebra and we have two possibilities. Let P_k be one of the two maximal parabolic subgroups containing the complexification $K_{\mathbf{C}}$ of the maximal compact subgroup K of G. Then $\operatorname{int} S(P) \neq \emptyset$ if and only if either $P \cap P_k$ or $P \cap \overline{P}_k$ contains a Borel subgroup. In the first case $S(P) = S(P_k)$ and $S(P) = S(P_k)^{-1}$ in the second case.

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