A CONVERSE OF CAUCHY'S THEOREM AND APPLICATIONS TO EXTREMAL PROBLEMS

BY

ARTHUR H. READ

University of St Andrews, Scotland

1. Introduction

1.1. In recent years many papers have been concerned with pairs of extremal problems which are conjugate in the sense that the extremal values are the same [1, 5, 6, 7, 11, 12, 13, 16]. The conjugacy is usually related to the conjugacy of the Lebesgue classes L_p and L_q where $p^{-1}+q^{-1}=1$; in the one problem one is maximizing an L_p norm, in the other minimizing an L_q norm. It is now well known that the conjugacy of such problems and the existence of extremals can be derived from the Hahn–Banach Theorem and related results. In this process the following converse of Cauchy's Theorem is of great assistance: If W is a region whose boundary C consists of a finite number of analytic Jordan curves, and if g is a bounded measurable function defined on C such that $\int_C g\,\omega=0$ for all differentials ω analytic in the closure of W, then g represents p.p. the boundary values of a function analytic in W. A proof of this theorem is given by Rudin [12] for the case of plane regions, and the theorem can be extended to Riemann surfaces. In the present paper a rather more general theorem of this type is proved for Riemann surfaces (Theorem 2.2 and its corollary), enabling a greater variety of extremal problems to be handled.

In studying the conjugate extremal problems it is convenient to consider separately the cases $1<p<\infty$, $p=1$ and $p=\infty$. The last two cases seem to have the more direct significance on Riemann surfaces, but special difficulties are apt to arise in their discussion. An account of the case in which the maximal problem is of type $p=1$ has been given in [11]. The present paper deals with the case in which the maximum problem is of type $p=\infty$, but its results are of greater variety owing to