TRANSLATION INVARIANT SPACES

To Karl Loewner

BY

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1. Let h denote the space of complex-valued square integrable functions $u(x)$ defined for x real which are zero for x negative. Let H denote the space of functions $U(z)$ which are Fourier transforms of functions in h. The space H is characterized by the one-sided

PALEY–WIENER THEOREM. (2) *Every function U in H can be extended as regular analytic to the upper half-plane, so that*

$$\int_{-\infty}^{\infty} U^*(i\tau+\sigma)\, U(i\tau+\sigma)\, d\sigma \leqslant \text{const},$$

for all τ positive. Conversely, the restriction to the real axis of any such function belongs to H.

For fixed τ, $U(i\tau+\sigma)$ is the Fourier transform of $e^{-x\tau}u(x)$; since $u(x)$ vanishes for negative x, the L_2 norm of $e^{-x\tau}\, u(x)$ decreases with increasing τ. So by Parseval's formula we have this

COROLLARY. *If U lies in H, its L_2 norm along the line* $\operatorname{Im} z = \tau$, $\tau \geqslant 0$, *decreases with increasing τ.*

The orthogonal complement of h with respect to the space of square integrable functions on the entire real axis is the space of square integrable functions which vanish for x positive. The Fourier transforms of these functions form the orthogonal complement $H^{\perp}$ of H. Functions in $H^{\perp}$ can be continued analytically into the lower halfplane. Also, it is easy to show that $H^{\perp}$ is the conjugate of H:

$$H^{\perp} = H^*$$

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(2) We denote the conjugate of a complex number by *; in section 4 where we deal with matrix valued functions the * denotes the adjoint.