A STAR DOMAIN WITH DENSEST ADMISSIBLE POINT SET NOT A LATTICE

BY

M. R. von WOLFF

St. Mary's Dominican College, New Orleans, U.S.A.(¹)

1. Introduction

A problem of wide interest in the geometry of numbers is the determination of the density of the closest packing of translates of a given body $\mathcal{K}$ in the plane. For $\mathcal{K}$ convex L. Fejes Toth [2] and C. A. Rogers [4] proved independently that any packing of translates of a convex body in the plane has a density not greater than the density of the best lattice packing.

The problem of packing convex bodies in the plane can be extended to the non-convex case in two distinct ways. One is to require that non-convex bodies be packed in a non-overlapping fashion. The other, the Minkowski-Hlawka type, allows overlapping of the bodies under conditions which highlight the relation between the critical lattice of a convex body and its best lattice packing. It is this latter type of packing which we consider in what follows.

Definitions

1. Let S be a star domain, symmetric about O. A set of points $\mathcal{D}$ is said to provide a *packing for* S if the domains $\{S+P\}_{P\in\mathcal{D}}$ have the property that no domain $S+P_0$ contains the center of another in its interior. We shall also say that $\mathcal{D}$ is an *admissible* point set for S.

2. The *density* of a lattice, $\mathcal{D}(\mathcal{L})$, is the reciprocal of its determinant.

3. Consider the square $|x|<t$, $|y|<t$. Let $A(t)$ denote the number of points of a set $\mathcal{D}$ in the square; then the *density* of $\mathcal{D}$ (denoted $\mathcal{D}(\mathcal{D})$, is defined as $\overline{\lim_{t\to\infty}}\frac{A(t)}{4t^2}$ ([5], p. 5).

That the Rogers' theorem does not hold generally for non-convex figures is shown by following example of a bounded star domain for which the densest packing is not a lattice.

2. Description of S

We take S to be the region defined in Fig. 1.

Let the point set $\mathcal{D}$ be the union of a lattice $\mathcal{L}_1$ of determinant 1 and $\mathcal{L}_2$, some trans-

(¹) This is part of the author's doctoral thesis at the University of Notre Dame (1961) under the direction of Professor Hans Zassenhaus.