THE MINIMUM OF A BINARY QUARTIC FORM (I).

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I. Introduction.

1. The question of the lower bound or minimum of an algebraic form $\varphi(x_1, \ldots, x_m)$ for integer values, not all zero, of the variables $x_1, \ldots, x_m$ is an important one and has attracted a great deal of attention for many years. The problem is a difficult one, however, and relatively few results are known.

Confining our attention to the case of forms with real coefficients in the two variables x, y, the results for the binary quadratic form are classical. If

$$\varphi(x, y) = ax^2+bxy+cy^2$$

is a binary quadratic of discriminant $D = b^2-4ac$, the result is that there exist integers x, y, not both zero, such that

$$|\varphi(x, y)| \leq k\sqrt{|D|}\,, \tag{1.1}$$

where $k = \frac{1}{\sqrt{3}}$ when $D < 0$ (that is, when the quadratic has complex roots[1]), and $k = \frac{1}{\sqrt{5}}$ when $D > 0$ (that is, when the quadratic has real roots). When $D = 0$ the lower bound is trivially found to be zero. These results are best possible, in the sense that the inequality (1.1) is no longer true for all forms of the type specified if the constant k is replaced by a smaller number.

Estimates for the lower bound of a binary cubic form were given many years ago by ARNDT (1) and HERMITE (2), but the best possible results were obtained only recently, by MORDELL (3). If now

[1] By the "roots" of a binary form $\varphi(x, y)$ we mean the roots of $\varphi(x, 1) = 0$.