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Regularity of a boundary having a Schwarz function

by

MAKOTO SAKAI

Tokyo Metropolitan University
Tokyo, Japan

In his book [5], Davis discussed various interesting aspects concerning a Schwarz function. It is a holomorphic function S which is defined in a neighborhood of a real analytic arc and satisfies $S(\zeta)=\bar{\zeta}$ on the arc, where $\bar{\zeta}$ denotes the complex conjugate of ζ.

In this paper, we shall define a Schwarz function for a portion of the boundary of an arbitrary open set and show regularity of the portion of the boundary. More precisely, let Ω be an open subset of the unit disk B such that the boundary $\partial\Omega$ contains the origin 0 and let $\Gamma=(\partial\Omega)\cap B$. We call a function S defined on $\Omega\cup\Gamma$ a Schwarz function of $\Omega\cup\Gamma$ if

(i) S is holomorphic in Ω,

(ii) S is continuous on $\Omega\cup\Gamma$,

(iii) $S(\zeta)=\bar{\zeta}$ on Γ.

We shall give a classification of a boundary having a Schwarz function. The main theorem, Theorem 5.2, asserts that there are four types of the boundary if 0 is not an isolated boundary point of Ω: 0 is a regular, nonisolated degenerate, double or cusp point of the boundary. Namely, one of the following must occur for a small disk B_δ with radius $\delta>0$ and center 0:

(1) $\Omega\cap B_\delta$ is simply connected and $\Gamma\cap B_\delta$ is a regular real analytic simple arc passing through 0.

(2 a) $\Gamma\cap B_\delta$ determines uniquely a regular real analytic simple arc passing through 0 and $\Gamma\cap B_\delta$ is an infinite proper subset of the arc accumulating at 0 or the whole arc. $\Omega\cap B_\delta$ is equal to $B_\delta\setminus\Gamma$.

(2 b) $\Omega\cap B_\delta$ consists of two simply connected components Ω_1 and Ω_2. $(\partial\Omega_1)\cap B_\delta$ and $(\partial\Omega_2)\cap B_\delta$ are distinct regular real analytic simple arcs passing through 0. They are tangent to each other at 0.