ON THE COMPLETENESS OF SOME SETS OF FUNCTIONS.

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1. Introduction.

A set of functions $\{\psi_n(x)\}$ is said to be complete in a space $L^p(a, b)$ $(1 \leq p \leq \infty)$, if $\int_a^b f(x)\,\overline{\psi}_n(x)\,dx = 0$ $(n = 1, 2, 3, \ldots)$ implies $f(x) \equiv 0$ when $f(x) < L^p(a, b)$. Let the differential equation

$$Ly = \lambda y, \quad L \equiv -\frac{d^2}{dx^2} + q(x) \tag{L}$$

together with linear homogeneous boundary conditions at the end-points of an interval (a, b) $(-\infty < a < b \leq +\infty)$ define a regular or singular boundary-value problem of a Sturm-Liouville type[1], whose eigenfunctions form a set, complete in $L^2(a, b)$. Then, in general, the set of squares on the eigenfunctions cannot be complete in $L^2(a, b)$ (for instance the set $\{\sin^2 nx\}$, belonging to (L) for $q(x) = 0$ and boundary conditions $y(0) = y(\pi) = 0$, has the completeness properties of the set $\{\cos 2nx\}$). In this paper some completeness properties of sets of eigenfunction-squares will be studied. The problems arose at the study of so-called inverse boundary-value problems, i. e. problems where the differential equation is to be determined from the knowledge of the spectrum and boundary conditions.[2]

The main results are, roughly speaking, the following.

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[1] In the sequel we use *S-L* as an abbreviation of Sturm-Liouville.

[2] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Acta math. 78 (1945). ——, Inverse Problems in the theory of Characteristic Values of Differential Systems, Dixième Congrès des Mathématiciens Scandinaves, Copenhague 1946. In these papers some results concerning eigenfunction-squares of regular *S-L* problems are contained.

34—48173. *Acta mathematica*. 81. Imprimé le 28 avril 1949.