ON TWO PROBLEMS CONCERNING LINEAR TRANSFORMATIONS IN HILBERT SPACE.

By

ARNE BEURLING

of UPPSALA.

Introduction.

1. Let H be a Hilbert space and T and T^* two adjoined transformations, both determined throughout H. Let Φ_λ be the set of eigenelements of T, corresponding to λ, i. e. the solutions $\varphi \neq 0$ of the equation $T\varphi = \lambda\varphi$; and Φ the sum of all Φ_λ. Firstly we assume that

(A) *the set Φ is fundamental on H.*

We shall denote by C_f and C_g^* the closed linear manifolds spanned by $\{T^n f\}_0^\infty$ and $\{T^{*n} g\}_0^\infty$, respectively; f, g being elements in H.

This study is devoted to two general problems concerning the transformations T and T^* which we shall call the extinction problem and the closure problem. We shall say that T has *an extinction theorem* if, for every $f \neq 0$, it is true that the manifold C_f contains at least one eigenelement $\varphi \neq 0$. In the case

$$f = \sum_{\nu=0}^{n} c_\nu \varphi_\nu, \quad \varphi_\nu \in \Phi_{\lambda_\nu},$$

where $\lambda_\nu \neq \lambda_\mu$ for $\nu \neq \mu$, it is obvious that all φ_ν belong to C_f. By (A), every f may certainly be approximated arbitrarily closely by linear combinations of eigenelements; but this does by no means imply that the extinction theorem is a consequence of (A).

By the closure problem we mean the characterizing of the elements g, for which $C_g^* = H$, by the behaviour of the scalar product (φ, g), when φ runs through Φ. From the relations

$$(\varphi_\lambda, T^{*n} g) = (T^n \varphi_\lambda, g) = \lambda^n (\varphi_\lambda, g), \quad n \geq 0,$$