AN ISOPERIMETRIC INEQUALITY FOR CLOSED CURVES CONVEX IN EVEN-DIMENSIONAL EUCLIDEAN SPACES*

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Introduction

1. The main theorem. A closed convex curve in the plane E_2 is usually defined as the boundary of a compact convex set.[1] Alternatively, if the curve is given in parametric form we could say that the curve is *convex* provided it never crosses a straight line more than twice. The second definition has the advantage of extending in a natural way to closed curves in an even-dimensional space E_{2n} as follows:

Let

$$C: \quad x_i = x_i(t), \quad (i = 1, \ldots, 2n;\ 0 \le t \le 2\pi), \tag{1}$$

where $x_i(t)$ are continuous functions of period 2π, be a closed curve in E_{2n}. We shall say that C is *convex in* E_{2n} provided that it never crosses a hyperplane more than $2n$ times. If C is convex in E_{2n} and spans the space E_{2n}, i.e. is not contained in a lower-dimensional flat space, then we shall say that C is *convex on* E_{2n}. It will be shown below (Article 5) that curves convex in E_{2n} are rectifiable. As an example of a curve convex on E_{2n} we mention the curve

$$C_0: \quad x_1 = \cos t,\ x_3 = \tfrac{1}{2}\cos 2t,\ \ldots,\ x_{2n-1} = \frac{1}{n}\cos nt, \tag{2}$$

$$x_2 = \sin t,\ x_4 = \tfrac{1}{2}\sin 2t,\ \ldots,\ x_{2n} = \frac{1}{n}\sin nt, \quad (0 \le t \le 2\pi).$$

Indeed, C_0 is convex in E_{2n}, for if $l(x_1, \ldots, x_{2n})$ is any linear function and if we substitute the x_i as defined by (2), we find that $l = T_n(t)$ is a real trigonometric

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[1] See [1], page 3, in the list of references at the end of this paper.